

# Decoherence-free subspace and disentanglement dynamics for two qubits in a common non-Markovian squeezed reservoir

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(Dated: July 23, 2010)

We study the non-Markovian entanglement dynamics of two qubits in a common squeezed bath. We see remarkable difference between the non-Markovian entanglement dynamics with its Markovian counterpart. We show that a non-Markovian decoherence free state is also decoherence free in the Markovian regime, but all the Markovian decoherence free states are not necessarily decoherence free in the non-Markovian domain. We extend our calculation from squeezed vacuum bath to squeezed thermal bath, where we see the effect of finite bath temperatures on the entanglement dynamics.

PACS numbers: 03.65.Yz, 03.67.Pp, 03.67.Mn

## I. INTRODUCTION

Entanglement is a remarkable feature of quantum mechanics, and its investigation is both of practical and theoretical significance. It is also viewed as a basic resource for quantum information processing [1], like entanglement-based quantum cryptography [2], quantum teleportation [3], dense coding [4] and cluster-state quantum computation [5]. Thus the real world success of these quantum information processing schemes relies on the longevity of entanglement in multiparticle quantum states. Entanglement is also related to the basic issue of understanding the nature of nonlocality in quantum mechanics [6, 7]. However, a quantum system used in quantum information processing inevitably interacts with the surrounding environment, which induces the quantum world into the classical world [8, 9]. The presence of decoherence in communication channels and computing devices degrades the entanglement when the particles propagate or computation evolves. The coupling of the quantum system with its surroundings and the consequent decay of entanglement motivate important questions such as to understand its sources, and possibly to find ways to circumvent it through different types of controlled environments.

The dynamics of open quantum systems, however, may be rather involved, mostly due to the complex structure of the environment interacting with the quantum system. Generally, the nonunitary evolution of the reduced density matrix of the system is obtained after taking partial trace over bath variables. In this process, some approximations are often made in the derivation of a master equation for the systems reduced density matrix. The most important approximations [8, 10] are the weak coupling or *Born approximation* assuming that the coupling between the system and the reservoir is small enough

to justify a perturbative approach, and the *Markov approximation* assuming that the correlation time of the reservoir is very short compared to the typical system response time so that the reservoir correlation function is assumed to be  $\delta$ -correlated in time. Although, the use of Markovian approximation is justified in a large variety of quantum optical experiments where entanglement has been produced, one should notice that non-Markovian effects are crucial, e.g., for high-speed quantum communication where the characteristic time of the relevant system become comparable with the reservoir correlation time, or if the environment is structured with a particular spectral density, e.g. for quantum systems (channels) embedded in solid-state devices, where memory effects are typically non-negligible. In these cases, the dynamics can be substantially different from the Markovian one. Due to their fundamental importance in quantum information processing and quantum computation, non-Markovian quantum dissipative systems have attracted much attention in recent years [8, 11–22], one of the main purposes of which in the long run is to engineer different types of (artificial) reservoirs, and couple them to the system in a controlled way. The non-Markovian features of system-reservoir interactions have made great progress, but the theory is far from completion, in particular how different kinds of non-Markovian environments influence the systems, and the difference between Markovian and non-Markovian system evolutions are still a subject that demands further investigations.

If the environment would act on the various parties the same way it acts on single systems, one would expect that a measure of entanglement, say the concurrence, would also decay exponentially in time. However, this is not always the case. Recently Yu and Eberly [23] showed that under certain conditions, the dynamics could be completely different and the quantum entanglement of a bipartite qubit system may vanish in a finite time. They called this effect “entanglement sudden death (ESD)”. This phenomenon of entanglement sudden death has been extensively studied [24] in the context of

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Markovian master equation. Entanglement dynamics for system of two qubits interacting with their local independent reservoir is completely different from that when they interact with the same common reservoir. We see that entanglement may not be destroyed by the interaction with the environment, and sometimes it persists at whatever be the temperature of the bath. A common reservoir indirectly couples the qubits, and there have been some suggestions [25] for creating as well as enhancing [26] entanglement between two or more parties by their collective interactions with a common environment. In the same vein, non-Markovian entanglement dynamics is fundamentally different from the Markovian one. With respect to the study of non-Markovian dynamics of two independent qubits, each locally interacting with its own reservoir, it was shown [18, 19] that although no interaction is present or mediated between the qubits, there is a revival of their entanglement following the ESD. The backaction of the non-Markovian reservoir [18, 19] is responsible for revivals of entanglement after sudden death. The dynamics of entanglement in two independent non-Markovian channels is shown [20] to be oscillating at high temperatures, whereas in the Markovian channel, entanglement was shown to decay exponentially. Appearance of sudden death and sudden birth of entanglement was also discussed [21] in common structured reservoirs.

Different schemes have been derived to remove the effects produced by the environment, for example, quantum error correction, decoherence-free subspace (DFS), dynamical decoupling and quantum Zeno effect. In the presence of the environment, the DFS is a set of all states which is not affected at all by the interaction with the bath. There were some proposals related to the use of DFS as the memory space for storing the quantum information [27]. Recently, decoherence free entanglement was studied for two two-level systems interacting with a common squeezed vacuum bath [28]. The Markovian entanglement dynamics of two two-level atoms that interact with a common squeezed vacuum reservoir was extensively studied [29]. The phenomenon of sudden death and revival of entanglement was investigated for the initial states that are very close to (as well as far from) the Markovian DFS. It was claimed that for states belonging initially to the DFS plane, the phenomenon of entanglement sudden death never occurs. However, if the initial state is away from the DFS plane, the sudden death shows up, followed by sudden revival of entanglement.

Several proposals to physically realize the squeezed reservoir were put forward in the literature, the simplest of which consists in considering a two-level atom immersed in a squeezed multimode radiation field [32–34]. Parkins *et al.* [35] showed how a two-level system can be coupled to an almost ideal squeezed vacuum by assuming an atom strongly interacting with a cavity field which is illuminated by finite-bandwidth squeezed light. It was also shown [37] how a squeezed environment can be obtained by means of a suitable feedback of the output signal corresponding to a quantum-nondemolition

(QND) measurement of suitable quadrature operators. In Ref. [36], the authors showed how to mimic the interaction of a two-level system with a squeezed reservoir by using a four-level atom interacting with circularly polarized laser fields. Assuming a strong decay of the two most excited levels, it was shown that the dynamics of the two ground atomic states is effectively similar to that of a two-level system interacting with a squeezed reservoir. A recent theoretical study [38] was made to generate squeezed reservoir for a two-level system by engineering the Hamiltonian of a  $\Lambda$ -type three-level atom interacting with a single cavity mode and laser fields with suitable intensity and detuning. Another experimental proposal [39] for two two-state atoms in a common squeezed reservoir was made employing quantum-reservoir engineering to controllably entangle the internal states of two atoms trapped in a high-finesse optical cavity. Using laser and cavity fields to drive two separate Raman transitions between stable atomic ground states, a system, corresponding to a pair of two-state atoms coupled collectively to a squeezed reservoir, could be realized.

In this paper, we will analyze the entanglement dynamics for two qubits interacting with a common squeezed reservoir in the non-Markovian regime. We see multiple cycles of entanglement sudden death and revival in the non-Markovian case, showing striking difference between the Markovian and non-Markovian entanglement dynamics. We extend our result for the finite temperature case where we see the non-Markovian entanglement oscillations gradually decreases as one increases the temperature. Finite temperature of the bath accelerates the phenomenon of ESD in general. We show that the Markovian decoherence free states remains invariant under finite temperature of the bath, and all states in the Markovian DFS plane is not necessarily decoherence free in the non-Markovian regime. Interestingly, the singlet state (which satisfy a more general DFS condition) is found to be decoherence free both in the Markov and non-Markov regime, and is also found to be robust against finite bath temperature. In Sec. II, we describe our model and present the non-Markovian master equation. In Sec. III we discuss the difference of DFS and entanglement dynamics between the Markovian and non-Markovian cases. In Sec. IV we present the numerical results and discuss our main observations, ending it with some concluding remarks in Sec. V.

## II. MODEL AND QUANTUM MASTER EQUATIONS

We consider a pair of two-level atoms (two qubits) coupled to a common non-Markovian thermal squeezed reservoir. The microscopic Hamiltonian of the system plus reservoir is given by

$$H = \omega_0 (\sigma_+^1 \sigma_-^1 + \sigma_+^2 \sigma_-^2) + \sum_k \omega_k b_k^\dagger b_k + H_I, \quad (1)$$

where the interaction Hamiltonian  $H_I$  has the form

$$H_I = \sum_k g_k S_+ b_k + g_k^* S_- b_k^\dagger. \quad (2)$$

Here  $S_+ = \sigma_+^1 + \sigma_+^2$  and  $S_- = \sigma_-^1 + \sigma_-^2$  are collective raising and lowering operators for the two-qubit system with  $\sigma_+^i = |1_i\rangle\langle 0_i|$ ,  $\sigma_-^i = |0_i\rangle\langle 1_i|$ , where  $|1_i\rangle$  and  $|0_i\rangle$  are up and down states of the  $i$ th qubit, respectively. Let us now proceed for the master equation for this two-qubit system interacting with a common squeezed thermal bath according to the Hamiltonian (1). We assume the factorized initial system-reservoir state with the initial state of the reservoir as a squeezed thermal equilibrium state given by

$$\rho_R(0) = \prod_k U(r_k, \theta_k) \rho_{th} U^\dagger(r_k, \theta_k), \quad (3)$$

$$U(r_k, \theta_k) = \exp \left( \frac{1}{2} \xi_k^* b_k^2 - \frac{1}{2} \xi_k b_k^{\dagger 2} \right), \quad (4)$$

$$\rho_{th} = \frac{\exp(-\beta \omega_k b_k^\dagger b_k)}{\text{Tr } \exp(-\beta \omega_k b_k^\dagger b_k)}, \quad (5)$$

where  $\beta = 1/KT$  with  $K$  being the Boltzman constant and  $T$  being the temperature, and we have introduced the unitary squeeze [8] operator  $U(r_k, \theta_k)$  with  $\xi_k = r_k e^{i\theta_k}$ .

The non-Markovian master equation in the interaction picture for the two-qubit reduced density matrix  $\rho(t)$  in the Born approximation can be calculated and written as:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \Delta(t) \{S_+ \rho S_- - \rho S_- S_+\} + \Delta^*(t) \{S_+ \rho S_- - S_- S_+ \rho\} \\ &\quad + \mu(t) \{S_- \rho S_+ - S_+ S_- \rho\} + \mu^*(t) \{S_- \rho S_+ - \rho S_+ S_-\} \\ &\quad + \alpha(t) \{2S_+ \rho S_+ - S_+ S_+ \rho - \rho S_+ S_+\} \\ &\quad + \alpha^*(t) \{2S_- \rho S_- - S_- S_- \rho - \rho S_- S_-\}. \end{aligned} \quad (6)$$

The time dependent coefficients appearing in the master equation are respectively given by

$$\Delta(t) = \int_0^t dt_1 \int_0^\infty d\omega J(\omega) N(\omega) e^{i(\omega_0 - \omega)(t-t_1)}, \quad (7)$$

$$\mu(t) = \int_0^t dt_1 \int_0^\infty d\omega J(\omega) [1 + N(\omega)] e^{i(\omega_0 - \omega)(t-t_1)} \quad (8)$$

$$\alpha(t) = \int_0^t dt_1 \int_0^\infty d\omega J(\omega) M(\omega) e^{i(\omega_0 - \omega)(t+t_1)}, \quad (9)$$

where

$$N(\omega) = n(\omega) [\cosh^2 r + \sinh^2 r] + \sinh^2 r, \quad (10)$$

$$M(\omega) = -\cosh r \sinh r e^{i\theta} [2n(\omega) + 1] \quad (11)$$

with the frequency independent resonant squeeze parameter  $r$  and the resonant phase  $\theta$ , and we have

written  $n(\omega) = 1/[\exp(\beta\omega) - 1]$  for the Planck distribution. The master equation (6) is valid for arbitrary temperature (provided that the Born approximation still holds) and the squeezed vacuum reservoir case is just the zero-temperature limit of it. In the zero-temperature limit,  $n(\omega) = 0$ , so,  $N(\omega) = N = \sinh^2(r)$ , and  $M(\omega) = -\cosh(r) \sinh(r) e^{i\theta} = -M e^{i\theta}$ , where  $M = \sinh(r) \cosh(r) = \sqrt{N(N+1)}$ . The non-Markovian character is contained in the time-dependent coefficients, which contain the information about the system-reservoir correlations. In the previous equations,  $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$  is the spectral density characterizing the bath, where the index  $k$  labels the different field mode of the reservoir with frequency  $\omega_k$ . We may consider any form of the reservoir spectral density. But for simplicity, here we consider an Ohmic squeezed bath [11, 12] with the spectral density given by

$$J(\omega) = \Gamma \omega \exp(-\omega^2/\omega_c^2), \quad (12)$$

where  $\omega$  is the frequency of the bath and  $\omega_c$  is the high-frequency cutoff and  $\Gamma$  is a dimensionless constant characterizing the interaction strength to the environment. For finite temperature Markovian case,  $n(\omega) = n(\omega_0)$ . Since we consider two qubits in a common bath, they will have to be quite near so that the interatomic separation is much smaller than a typical wavelength of the bath, i.e., the length scale of the resonant wavelength  $\lambda_0 = \hbar c/\omega_0$  in the model [40], where  $c$  is the wave speed of the bath. Thus we do not consider here the effect of qubit size, also the spatial dependence on qubit-environment coupling strength [40, 41] ( $g_k$  is assumed to be position-independent). We also assume that there is no direct interaction between the qubits except the indirect coupling through the common environment.

### III. DIFFERENCE BETWEEN MARKOVIAN AND NON-MARKOVIAN DYNAMICS

Our next aim is to show the difference between the Markovian and non-Markovian entanglement dynamics for this system-reservoir model. The Markovian master equation, in the interaction picture, for two two-level systems interacting with a broadband squeezed vacuum bath (at zero temperature) is well studied [28–30] and is given by

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{1}{2} \gamma (N+1) (2S_- \rho S_+ - S_+ S_- \rho - \rho S_+ S_-) \\ &\quad + \frac{1}{2} \gamma N (2S_+ \rho S_- - S_- S_+ \rho - \rho S_- S_+) \\ &\quad - \frac{1}{2} \gamma M e^{i\theta} (2S_+ \rho S_+ - S_+ S_+ \rho - \rho S_+ S_+) \\ &\quad - \frac{1}{2} \gamma M e^{-i\theta} (2S_- \rho S_- - S_- S_- \rho - \rho S_- S_-) \end{aligned} \quad (13)$$

where  $\gamma = \pi \Gamma \omega_0$ . Using this Markovian master equation (13), the entanglement dynamics (the phenomenon of sudden death and revival of entanglement) of a pair of

two-level atoms has been extensively studied [29]. The DFS for this model (Markov approximation, broadband squeezed vacuum) is found in Ref. [28], and we call it Markovian DFS.

The main result of the present paper will rotate around the discussion of DFS and entanglement dynamics according to the Markovian master equation (13) and its non-Markovian counterpart (6), showing striking difference between them. This Markovian master equation (13) can be written in an explicit Lindblad form using only one Lindblad operator [28]:

$$\frac{\partial \rho}{\partial t} = \frac{\gamma}{2} (2L\rho L^\dagger - \rho L^\dagger L - L^\dagger L\rho), \quad (14)$$

where

$$L = \sqrt{N+1} S_- - \sqrt{N} \exp\{i\theta\} S_+. \quad (15)$$

In this case, the DFS [42] is composed of all eigenstates of  $L$  with zero eigenvalues. The two orthogonal vectors in the DFS plane for this Markovian evolution are [28]

$$|\phi_1\rangle = \frac{1}{\sqrt{N^2 + M^2}} (N|11\rangle + M e^{-i\theta}|00\rangle), \quad (16)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \quad (17)$$

One can also define the states  $|\phi_3\rangle$  and  $|\phi_4\rangle$  orthogonal to  $\{|\phi_1\rangle, |\phi_2\rangle\}$  plane:

$$|\phi_3\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \quad (18)$$

$$|\phi_4\rangle = \frac{1}{\sqrt{N^2 + M^2}} (M|11\rangle - N e^{-i\theta}|00\rangle). \quad (19)$$

It is important to mention here that the state  $|\phi_1\rangle$ , Eq. (16), is a decoherence free state only for the Markovian evolution (13), but for a general non-Markovian dynamics  $|\phi_1\rangle$  is not decoherence free. A more general discussions on DFS condition was discussed in Refs.[42, 43]. It is important to note that decoherence is the result of the entanglement between system and bath caused by the interaction term  $H_I$ , Eq. (2), of the Hamiltonian (1). In other words, if  $H_I = 0$  then system and bath are decoupled and evolve independently and unitarily under their respective Hamiltonians  $H_S$  and  $H_B$ . Clearly, then, a sufficient condition for decoherence free dynamics is that  $H_I = 0$ . However, since one cannot simply switch off the system-bath interaction, in order to satisfy this condition, it is necessary to look for special subspace (say,  $\mathcal{H}$ ) of the full system Hilbert space such that the system evolves in a completely unitary fashion on that subspace  $\mathcal{H}$ . As shown first by Zanardi and Rasetti [43], such a subspace is found by assuming that there exists a set of degenerate eigenvectors of the system coupling operators  $S_\pm$  in the system-reservoir interaction Hamiltonian. In our case, focusing on the form of interaction Hamiltonian  $H_I$  given by Eq. (2), it is clear that the DFS is made up of those states  $|\psi\rangle$  satisfying [42]

$$S_\pm|\psi\rangle = 0. \quad (20)$$

The singlet state, Eq. (17), is one which satisfies (20) with vanishing total angular momentum. Hence the singlet state  $|\psi\rangle = |\phi_2\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$  is a decoherence free state for this type of Hamiltonian. This is why the singlet state  $|\phi_2\rangle$  is decoherence free both for Markovian (13) and non-Markovian (6) dynamics at any temperature. This result is also confirmed by the numerical calculations shown in Sec. IV.

We note here that the qualitative characteristics of the non-Markovian entanglement dynamics with oscillations and sudden deaths and revivals (shown in Sec. IV) may not be specific to the squeezed reservoir, but the quantitative feature of this dynamics will depend on various properties of the reservoir, such as spectral density, squeezing, and temperature. Nevertheless, it is important to mention here that the Markovian master equation for two two-level system interacting with a common (*unsqueezed*) heat bath is

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \frac{1}{2}\gamma(N+1)(2S_-\rho S_+ - S_+ S_- \rho - \rho S_+ S_-) \\ & + \frac{1}{2}\gamma N(2S_+\rho S_- - S_- S_+ \rho - \rho S_- S_+). \end{aligned} \quad (21)$$

In this case, the DFS is composed of common eigenstates of the Lindblad operators  $L_\pm = S_\pm$  with zero eigenvalues. The singlet state  $|\phi_2\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$  satisfies this condition. So, the singlet state  $|\phi_2\rangle$  is the only state which is decoherence free for an unsqueezed common reservoir. Interestingly, we see that for an unsqueezed reservoir, DFS calculated from the Lindblad operators  $L_\pm = S_\pm$  and that obtained from the system-bath interaction (Zanardi and Rasetti criterion [42, 43]) Hamiltonian  $H_I$ , are the same. Hence there will be no difference between the Markovian and non-Markovian entanglement dynamics for this DFS state. On the other hand, for the squeezed reservoir, one can have infinitely many Markovian DFS states  $|\phi_1\rangle$ , Eq. (16), just by varying continuously the squeeze parameters  $\theta$  and  $r$ . In this case, squeezing plays an important role in showing difference between the Markovian and non-Markovian entanglement dynamics for these Markovian DFS states. This is quantitatively shown in Fig. 3 in the next section. Now, in order to study the sudden death and revival of entanglement of two qubits in this common squeezed bath (both in the Markovian and non-Markovian regimes), we consider as initial states of the form [28, 29]

$$|\Psi_1\rangle = \epsilon|\phi_1\rangle + \sqrt{1-\epsilon^2}|\phi_4\rangle, \quad (22)$$

$$|\Psi_2\rangle = \epsilon|\phi_2\rangle + \sqrt{1-\epsilon^2}|\phi_3\rangle, \quad (23)$$

where  $\epsilon$  is a variable amplitude of one of the states belonging to the Markovian DFS plane  $\{|\phi_1\rangle, |\phi_2\rangle\}$ . We would like to study the effect of varying  $\epsilon$  ( $0 \leq \epsilon \leq 1$ ) on the sudden death and revival of entanglement for these initial states. We calculate the time-evolved two-qubit density matrix for the initial states  $|\Psi_1\rangle$  and  $|\Psi_2\rangle$  using the Markovian (13) as well as the non-Markovian (6)

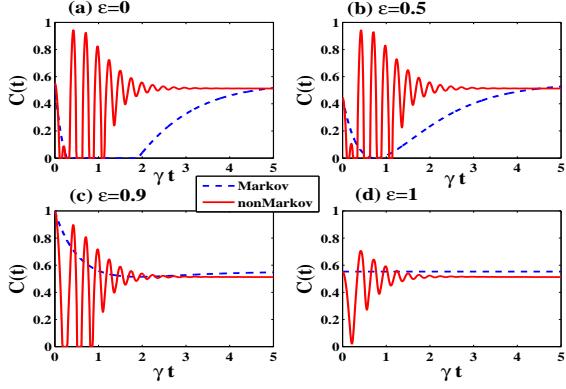


FIG. 1. (Color online) Non-Markovian (solid lines) and Markovian (dashed lines) time evolution of the concurrence for  $|\Psi_1\rangle$  as an initial state with varying  $\epsilon$ : (a)  $\epsilon = 0$ , (b)  $\epsilon = 0.5$ , (c)  $\epsilon = 0.9$ , and (d)  $\epsilon = 1$ . The values of the other parameters are  $r = 0.31$ ,  $\theta = 0$ ,  $\gamma = 1$ , and  $\omega_c = \omega_0 = 1$ . The number of ESD and revivals in the non-Markovian case increases. The state  $|\phi_1\rangle$  ( $|\Psi_1\rangle$  with  $\epsilon = 1$ ) no longer remains decoherence free for the non-Markovian evolution although it is decoherence free in the Markovian case.

master equations. The various components of the time-dependent density matrix depend on the initial states as well as on the squeezing parameters.

#### IV. NUMERICAL RESULTS

We calculate numerically the time evolution of the density matrix according to Eqs. (13) and (6) and we choose the Wootters entanglement measure [44], the concurrence  $C(t)$ , defined for the time-evolved two-qubit density matrix  $\rho(t)$  as

$$C(t) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \quad (24)$$

where  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  are the eigenvalues of the matrix  $\rho_c = \rho(t)(\sigma_y \otimes \sigma_y)\rho^*(t)(\sigma_y \otimes \sigma_y)$  in descending order. The entanglement dynamics (in both the Markovian and non-Markovian regimes) is shown in Fig. 1 for two qubits initially in the state  $|\Psi_1\rangle$ . The state  $|\Psi_1\rangle$  is a superposition of two states  $|\phi_1\rangle$  (belonging to the Markovian DFS) and its orthogonal  $|\phi_4\rangle$ . We vary  $\epsilon$  between 0 and 1 for fixed values of the parameters  $r = 0.31$  and  $\theta = 0$  as in Ref.[29]. Let us recapitulate the observation made in Ref.[29] about the Markovian dynamics of entanglement for the state  $|\Psi_1\rangle$  in the parameter interval  $0 \leq \epsilon < 0.5$  [see also the curve in dashed line in Fig. 1(a)], where it was shown that the initial entanglement decays to zero in a finite time  $t_d$ . Then after a finite period of time during which the concurrence stays null, it revives at a later time  $t_r$ , then reaching asymptotically its steady-state value. For this Markovian case, it was also observed that this death and revival cycle happens only once for the initial state  $|\Psi_1\rangle$ . For  $\epsilon = 0.5$ , the entanglement dies and revives simultaneously and eventually goes to

its steady-state value [see also the curve in dashed line in Fig. 1(b)]. For  $0.5 \leq \epsilon < 1$ , no entanglement sudden death was found [29] in the Markovian dynamics [see also the curve in dashed line in Fig. 1(c)]. Finally, when  $\epsilon = 1$ ,  $|\Psi_1\rangle = |\phi_1\rangle$  is a decoherence-free state so that the concurrence remains constant with time [see also the curve in dashed line in Fig. 1(d)]. This was the picture for the Markovian master equation (13). Now we compare this Markovian case with the non-Markovian entanglement dynamics according to Eq.(6) for the initial state  $|\Psi_1\rangle$ .

In the non-Markovian case for  $\epsilon = 0$ , we see from the curve in solid line in Fig.1(a) that the initial entanglement decays to zero in a finite time showing sudden death of entanglement and then it revives again, following four successive death and revival, and finally the steady state value of the concurrence is reached at large times. Whereas in the case of Markovian dynamics, we see only one death and revival. It is also important to note that the entanglement sudden death occurs much faster in the non-Markovian case compared to the Markovian one. The time gap between adjacent death and revival is small (that is the rate at which the death and revival occurs are very fast) compared to its Markovian counterpart. The striking difference shown in Fig.1(a) between the Markov and non-Markov entanglement dynamics is that in the non-Markovian case, the entanglement is nonzero (showing three revival cycle) in a time window when the concurrence remains null in the case of Markovian dynamics. Similar multiple death and revival cycle is observed for the state with  $\epsilon = 0.5$  in the non-Markovian case showing a clear departure from the Markovian dynamics. When  $0.5 \leq \epsilon < 1$ , that is when we get closer to the Markovian DFS, it was shown [29] that the whole phenomenon of sudden death and revival disappears for the initial state  $|\Psi_1\rangle$ . Contrary to that, we see [for the initial state  $|\Psi_1\rangle$  with  $\epsilon = 0.9$  in Fig.1(c)] clear sudden death and revival for this range of  $\epsilon$  as well in the *non-Markovian* case.

Fig. 2 shows the dynamical behavior of the entanglement in terms of the concurrence for the initial state  $|\Psi_2\rangle$ . The state  $|\Psi_2\rangle$  is a superposition of two states  $|\phi_2\rangle$  (belonging to the Markovian DFS) and its orthogonal  $|\phi_3\rangle$ . We again vary  $\epsilon$  between 0 and 1 for fixed values of the parameters  $r = 0.31$  and  $\theta = 0$ . For the initial state  $|\Psi_2\rangle$ , we also see multiple cycles of death and revival of entanglement in the non-Markovian regime (in the parameter interval  $0 \leq \epsilon < 1/\sqrt{2}$ ) showing completely different behavior from its Markovian counterpart [29]. When  $1/\sqrt{2} \leq \epsilon < 1$ , no sudden death was observed for the state  $|\Psi_2\rangle$  in the Markov regime whereas we see clear sudden death in this case also for the non-Markovian case [see Fig. 2(d)].

Finally we emphasize again that the state  $|\phi_1\rangle$  (the state  $|\Psi_1\rangle$  with  $\epsilon = 1$ ) is not a decoherence free state for non-Markovian master equation (6) although it belongs to the DFS for Markovian master equation (13) [see Fig. 1(d)]. The Markovian and non-Markovian entanglement dynamics for the Markovian DFS states  $|\phi_1\rangle$ , Eq. (16), with varying squeeze parameters  $r$  and  $\theta$  is

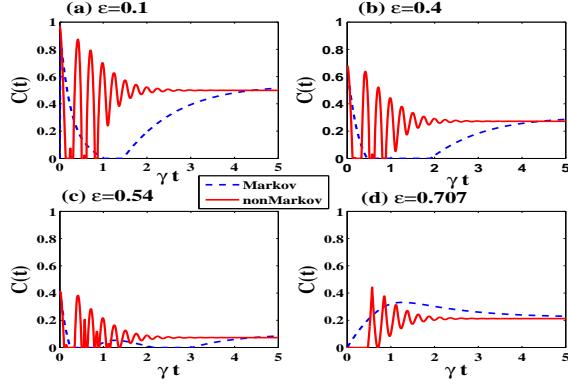


FIG. 2. (Color online) Non-Markovian (solid lines) and Markovian (dashed lines) time evolution of the concurrence for  $|\Psi_2\rangle$  as an initial state with varying  $\epsilon$ : (a)  $\epsilon = 0.1$ , (b)  $\epsilon = 0.4$ , (c)  $\epsilon = 0.54$ , and (d)  $\epsilon = 0.707$ . The values of the other parameters are  $r = 0.31$ ,  $\theta = 0$ ,  $\gamma = 1$ , and  $\omega_c = \omega_0 = 1$ . The number of ESD and revivals in the non-Markovian case increases. The state  $|\phi_2\rangle$  ( $|\Psi_2\rangle$  with  $\epsilon = 1$ ) remains decoherence free for both the Markovian and non-Markovian evolution.

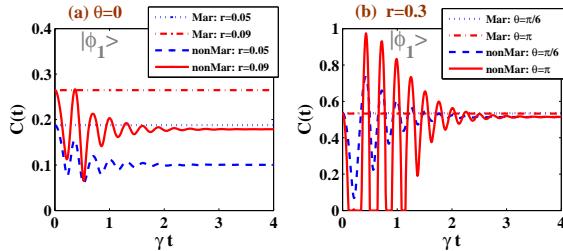


FIG. 3. (Color online) Non-Markovian entanglement dynamics for various Markovian DFS states  $|\phi\rangle$  with varying squeeze parameters  $r$  and  $\theta$ : (a)  $r = 0.05$  and  $r = 0.09$  with  $\theta = 0$ , (b)  $\theta = \pi/6$  and  $\theta = \pi$  with  $r = 0.3$ . The values of the other parameters are  $\gamma = 1$ , and  $\omega_c = \omega_0 = 1$ .

shown in Fig. 3. We consider in Fig. 3 two separate cases (a)  $r = 0.05$ ,  $r = 0.09$  with  $\theta = 0$  and (b)  $\theta = \pi/6$ ,  $\theta = \pi$  with  $r = 0.3$ . We find that the initial entanglement and final asymptotic entanglement at large times (Markov and non-Markov) for the initial Markovian DFS states, Eq. (16), depend on  $r$  not on  $\theta$ . One can observe from Fig. 3(b) that when the squeeze parameter  $r$  is fixed and the phase  $\theta$  is varied, the Markovian concurrence curves do not vary and they remain the same and overlap. This is because when  $r$  is varied, the values of  $N$  and  $M$  in Eq. (16) change considerably and so do the resultant Markovian DFS states, while different values of  $\theta$  change only the relative phase between  $|11\rangle$  and  $|00\rangle$  in the resultant Markovian DFS states of Eq. (16). Nevertheless, one can see that these Markovian DFS states are not decoherence-free in the non-Markovian regime and the characteristics of this non-Markovian oscillations of concurrence depend on the squeeze parameters  $r$  and  $\theta$ .

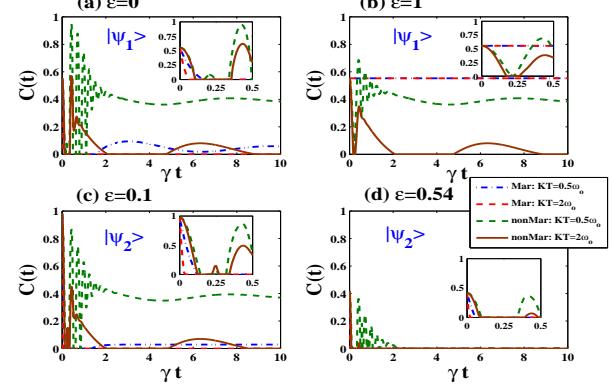


FIG. 4. (Color online) Non-Markovian and Markovian time evolution of the concurrence at finite temperatures for  $|\Psi_1\rangle$  as an initial state with (a)  $\epsilon = 0$  and (b)  $\epsilon = 1$ , and for  $|\Psi_2\rangle$  as an initial state with (c)  $\epsilon = 0.1$  and (d)  $\epsilon = 0.54$ . The values of the other parameters are  $r = 0.31$ ,  $\theta = 0$ ,  $\gamma = 1$ , and  $\omega_c = \omega_0 = 1$ . The insets show the entanglement dynamics in the short-time region. The non-Markovian entanglement oscillations gradually disappear as one increases the temperature. The state  $|\phi_2\rangle$  ( $|\Psi_2\rangle$  with  $\epsilon = 1$ ) remains decoherence free both in the Markov and non-Markov regime at any temperature of the reservoir. But the state  $|\phi_1\rangle$  ( $|\Psi_1\rangle$  with  $\epsilon = 1$ ) that is decoherence free at any reservoir temperature in the Markovian case no longer remains decoherence free for the non-Markovian evolution.

On the other hand, the state  $|\phi_2\rangle$  (the state  $|\Psi_2\rangle$  with  $\epsilon = 1$ ) is decoherence free both for Markovian and non-Markovian master equations. This was verified by obtaining a straight line  $C(t) = 1$ , showing that the entanglement remains constant for this state at zero temperature of the reservoir.

Next, we go to the finite-temperature case for which  $N(\omega)$  and  $M(\omega)$  are given by Eqs.(10) and (11). To show the effect of temperature on the Markovian evolution let us focus on the entanglement dynamics of two specific initial states ( $|\Psi_1\rangle$  with  $\epsilon = 0$ , and  $|\Psi_2\rangle$  with  $\epsilon = 0.1$ ). From the zero-temperature Markovian dynamics of these two states [see Fig. 1(a) and Fig. 2(a)] we see that at long time the entanglement finally saturates to a finite value after which no death of entanglement occurs, whereas at finite temperatures (say,  $KT > 2\omega_0$ ) Markovian dynamics of these two states show a complete death of entanglement at long-time limit [see Figs. 4(a), 4(c) and their insets]. From Fig. 2(a) we see for the zero-temperature case, Markovian entanglement sudden death occurs at relatively slower rate whereas in the finite-temperature case [Fig. 4(c)] ESD rate is much faster. The difference between the zero-temperature entanglement dynamics and the finite-temperature entanglement dynamics is more transparent in the non-Markovian case. The non-Markovian entanglement dynamics at zero temperature shows several death and revival cycles in short time limit while there is only one or two death-revival cycles following a decay of entanglement for the finite temperature (say,  $KT > 2\omega_0$ ) non-Markovian case. In the non-

Markovian evolution (Fig. 4), as one gradually increases the temperature, the number of death and revival cycles of the concurrence reduces and the effect of temperature on concurrence in general is seen as decaying in nature (concurrence asymptotically reaches to its zero value at large times) with an exception to the DFS. That is in the long time limit, the non-Markovian entanglement saturates to a certain nonzero value at low temperature, while at high temperatures ( $KT > 2\omega_0$ ) the concurrence finally decays to zero in the long-time limit. Note also that the concurrence disappear more quickly at finite temperatures for both Markovian and non-Markovian cases. It is seen that the death time for the Markovian dynamics is always less compared to the non-Markovian dynamics at a given finite temperature. We have seen that the state  $|\phi_2\rangle$  (the state  $|\Psi_2\rangle$  with  $\epsilon = 1$ ) remains decoherence free both in the Markov and non-Markov regime at any finite temperature of the reservoir. This was verified by obtaining a straight line  $C(t) = 1$ , showing that the entanglement remains constant for this state  $|\phi_2\rangle$  at any arbitrary temperature of the reservoir. It is quite striking that the state  $|\phi_1\rangle$  shows entanglement sudden death at finite temperature in the non-Markovian dynamics, althouth it remains a DFS at finite temperature for the Markovian case [see Fig. 4(b)]. We have also checked numerically that non-Markovian oscillations of entanglement gradually diminishes as one increases  $N$  defined in (10), for fixed values of other parameters.

## V. CONCLUSION

In summary, we study the non-Markovian entanglement dynamics of two qubits in a common squeezed bath. We consider the initial two-qubit states which are very close to (as well as far from) the Markovian DFS for this system. We see (Fig.1 and Fig.2) multiple cycles of entanglement sudden death and revival in the non-Markovian case, showing striking difference between the Markovian and non-Markovian entanglement

dynamics. We also observe that a non-Markovian decoherence free state (for example,  $|\phi_2\rangle$ ) remains decoherence free in the Markovian regime, but all the Markovian decoherence free states (for example,  $|\phi_1\rangle$ ) are not necessarily decoherence free in the non-Markovian domain [Fig.1(d)]. Finally, we extend our result for the finite-temperature case where we see the non-Markovian entanglement oscillations gradually decreases as one increases the temperature. We found that the Markovian decoherence-free states remain invariant under finite temperatures [Fig.4(b)]. We also see from Fig.4 that the finite temperature of the bath accelerates the phenomenon of entanglement sudden death for the non-decoherence-free entangled initial states. Interestingly, the state  $|\phi_2\rangle$  is found to be decoherence free both in the Markov and non-Markov regime, and is also robust against finite bath temperatures. There were considerable number of papers in recent literature [28, 29, 31] dealing with the squeezed reservoir. Also, there were proposals [32–36, 38, 39] for physical realizations of environments that mimic or generate Markovian squeezed bath. One can suggest in future experimental schemes to test and observe various important issues related to non-Markovian dynamics by physically engineering the environment. Hence, it will be interesting to see if our prediction of the non-Markovian effects can be verified in actual experiments.

## ACKNOWLEDGMENTS

We would like to acknowledge support from the National Science Council, Taiwan, under Grant No. 97-2112-M-002-012-MY3, support from the Frontier and Innovative Research Program of the National Taiwan University under Grants No. 97R0066-65 and No. 97R0066-67, and support from the focus group program of the National Center for Theoretical Sciences, Taiwan. We are grateful to the National Center for High-performance Computing, Taiwan, for computer time and facilities.

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